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RESEARCH ARTICLE

Who controls Allianz?

Measuring the separation of dividend and control rights under cross-ownership among firms

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Abstract With cross-ownership among firms a separation between dividend rights on common stocks and voting rights may occur. This paper proposes a method to trace control rights in a company that is based purely upon accounting identities and the underlying data on cross-ownership relations among firms and privately held shares. Examples show that under cross-ownership, control and ownership of dividend rights may be entirely separated, and multiple equilibria may exist in such economies. The proposed methodology is then applied to the conglomerate around the German 'Allianz' group.

Keywords Cross ownership · Control rights · Corporate governance · Voting · Allianz

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1 Introduction

Noren Inc. is a stock market listed company. The two largest blocks of single held shares amount to 25% each. One of these blocks is held by Crooks Ltd., the other by Rogue & Co. None of the three companies has a majority shareholder. The Chief Executive Officer (CEO) of the three firms is Mr. Len Kay, the champion among CEOs when it comes to salaries. How come?

Crooks Ltd. holds 50% in Rogue & Co. Likewise, Rogue & Co. owns 50% of Crooks Ltd. Mr. Kay owns one share in each of the latter two companies, and one in Noren Inc. When it comes to shareholder voting on the CEO (and his salary) for Crooks Ltd., Mr. Kay and Rogue & Co. vote for Mr. Kay and the maximum salary. Since this coalition amounts to 50% plus one share in Crooks Ltd., Mr. Kay gets established. For Rogue & Co. the shares held by Crooks Ltd. and Mr. Kay ensure that here too Mr. Kay gets paid a generous salary. Finally, when it comes to Noren Inc., the votes cast by Crooks Ltd., Rogue & Co., and Mr. Kay amount to 50% plus one share and establish Mr. Kay as the CEO at the best salary ever. Any move to unseat Mr. Kay would be doomed to fail.

The only expenses that Mr. Kay faces in building his empire is the price of the share in Noren Inc. and of the shares in each of Crooks Ltd. and Rogue & Co. Since the market value of the latter shares amounts to approximately one quarter of the value of Noren Inc., his expenses are equivalent to one and a half shares in Noren Inc.

Cross-ownership among firms has been widely recognized as a device to drive a wedge between ownership of dividend rights and control rights in a company (see e.g. La Porta et al. 1999; Shleifer and Vishny 1997 for a survey on corporate governance, or Berle and Means 1932 for a first analysis). That it is also empirically important is well documented. The 20 largest publicly owned firms in each of the 27 wealthiest countries are studied (see La Porta et al. 1999) and it is found that controlling shareholders often hold cash-flow rights considerably below their control rights, mostly due to pyramidal ownership. In addition, about 40% of the largest publicly traded corporations in East-Asia are controlled by pyramids (see Claessens et al. 2000). In North America and Europe cross-ownership among firms appears to be less prevalent and to coexist with other means of control, like dual-class shares (see Faccio et al. 2001; Faccio and Lang 2002). Still, some of the largest cases of cross-ownership reside in Europe, in particular around the Allianz insurance company in Germany.

Pyramids appear to be used to "tunnel" resources to the pockets of the controlling shareholder (see e.g. La Porta et al. 2002; Johnson et al. 2000; Bae et al. 2002 on Korean "chaebols," or Bertrand et al. 2002 on Indian firms). However, also the opposite can occur: controlling shareholders may "prop up" money-losing firms within a group to secure future tunneling opportunities (see Friedman et al. 2003; or Bae et al. 2002). Both tunneling and propping provide potential explanations for the existence of cross-ownership (see Wolfenzon

1999; Obata 2001; Riyanto and Toolsema 2004, or Almeida and Wolfenzon 2004).

Other explanations hold that buying control in a company, without buying significant claims to its returns, may be due to a desire to diversify risks (see Hansen and Lott 1996) or contracting under asymmetric information (Krasa and Villamil 2000). Or cross-ownership may be a vehicle to enforce collusion between firms under imperfect competition (see Macho-Stadler and Verdier 1991; Bolle and Güth 1992; Spagnolo 1998; Gilo et al. 2006).

There is also a literature that studies the value of control by looking at the premium over non-voting shares at which voting shares trade, for companies with dual-class shares (see Zingales 1994 for Italy, Rydqvist 1996 for Sweden, or Nenova 2003 and Modigliani and Perotti 1997 for cross-country comparisons). Invariably, these papers find a significant positive value for the premium, though of different magnitude across countries. If voting power confers value, it becomes important to identify who may exercise control, even if it is through a complex web of cross-holdings.

This paper does not aspire to provide a full explanation for cross-ownership, nor what distinguishes pyramids from other forms of cross-ownership. Instead, we address the accounting issue of how to identify controlling shareholders in arbitrary structures of cross-holdings and how to account for dividend rights.

Earlier studies in corporate finance have not made much of a distinction between ownership and control. Since La Porta et al. (1999), however, researchers have started tracing back control to some "ultimate" controlling shareholder. The idea is to look beyond a company's direct owners, i.e., to trace the owners of a company's owners, their owners, and so on.

The measures for ownership and control in the literature have largely relied on using pyramids as a template. In a pyramid a chain of firms is constructed so as to control voting rights in a target company (see Sect. 2 for a formal treatment). For such pure pyramids the product of shares held along the chain provides a measure of ownership rights. Likewise, for this particular ownership structure, the minimum of the shares held along the control chain proxies control rights.

Accordingly, if an investor "…owns a fraction x of the shares of corporation X, which owns a fraction y of the shares in corporation Y, which owns a fraction z of the shares in Z, he owns a fraction xyz of the shares of Z. However, his share of the control rights of Z via this control chain can be measured by its weakest link, i.e., the minimum of x, y, and z. For example, an investor who owns 50% of the shares of X, which owns 40% of the shares of Y, which owns 30% of the shares in Z, has 6% of the ownership rights of Z, but 30% of its control rights." (Faccio et al. 2001). Note that it is crucial in this numerical example that the investor owns 50% of company X, because otherwise he could not control the shares that X holds in Y, nor those that Y holds in company Z.

Applying these measures in the initial fictitious 'Noren Inc.' example, however, shows that for more complicated cross-ownership the product of shares and the minimum share along the chain are insufficient statistics, for ownership and control respectively. Since in the fictitious example Mr. Kay owns no significant fraction in any of the three companies, his control rights would be negligible according to the above computations. Yet, we have argued that Mr. Kay has full control over all three companies. Similarly, as the example is not a pyramid, the computation of ownership rights needs adjustment.

The case of Allianz illustrates that such non-pyramidal cross-ownership relations do arise empirically. In that case ownership is mostly both ways and the resulting structure resembles more a spider-web than a chain. This requires a more general measure of ownership rights on the one hand and control rights on the other.

As for control rights, this paper presumes that there does exist a controlling shareholder, or a group of controlling shareholders, for each company. To determine this individual or group we argue theoretically that holding the largest block of voting rights—directly and indirectly—constitutes a necessary and sufficient condition for control (see the 'dominant shareholder theorem' below). The problem of identifying controlling shareholders then boils down to assigning firm-held (voting) shares to private investors in such a way that an investor gets assigned the shares of company X held by company Y if and only if he controls a relative majority of voting rights in corporation Y, either directly or indirectly. Clearly, this calls for a simultaneous reassignment of firm-held shares for the whole economy. Effectively, it becomes a solution of a simultaneous equation system that takes account of all cross-ownership relations in the economy.

Take, for instance, the initial fictitious Noren Inc. example. Suppose that some investor, say, Mr. Kay, is found to hold directly and indirectly the largest block of voting stocks in Crooks Ltd. Since that puts him in control of Crooks Ltd., Mr. Kay gets assigned the shares that Crooks Ltd. holds in other companies. This concerns in particular the shares that Crooks Ltd. holds in Rogue & Co. and in Noren Inc. As a consequence, Mr. Kay now holds at least 50% of the voting shares in Rogue & Co. Since that makes him a relative majority owner of Rogue & Co., he gets assigned the shares that Rogue & Co. holds in other companies, in particular those in Crooks Ltd. and in Noren Inc. At this point Mr. Kay controls a majority in all three companies. This is so, because he has at least the 50% of Crooks Ltd. in Rogue & Co. to control Rogue & Co., at least the 50% of Rogue & Co. in Crooks Ltd. to control Crooks Ltd., and the 25% of Crooks Ltd. plus the 25% of Rogue & Co. in Noren Inc., i.e. a total of 50%, to control the target Noren Inc. At this point no more shares can be reassigned from firms to shareholders. This terminal assignment of voting shares, therefore, constitutes the share distribution on which the identification of controlling shareholders is based—in the case at hand it is Mr. Kay for all three companies.

Technically, this measure for control rests upon the construction of *control coefficients* that identify which investor exercises control over a particular firm. These control coefficients are used to reassign firm-held shares to private investors and, thereby, determine *control assignments* that identify who controls which firms simultaneously for the whole economy.

Since the underlying equation system is highly nonlinear, it comes as no surprise that a given economy may possess multiple control assignments. For instance, in the fictitious 'Noren Inc.' example above either Mr. Kay may control all companies, or a second investor, who owns the majority of dividend rights in all companies, may be the controlling shareholder of all three firms. This already illustrates that the notion of control assignments also captures control over a company without ownership. Accordingly, the notion of 'management control' assumes a precise meaning here: a company is *management controlled* (at some control assignment) if it can be controlled by an agent without ownership in dividend rights.

The determination of ownership in dividend rights that we propose here is also based on a simultaneous consideration of the whole economy (see the 'imputed shares' defined in Sect. 2). It is implied by the solution of the accounting identities for all firms, as all dividends are ultimately paid out to private investors. [This measure of ownership rights also arises in studies on tacit collusion under cross-ownership; see, in particular, Bolle and Güth (1992), Spagnolo (1998), and Gilo et al. (2006). A measure of cross-ownership effects that constitutes a first-order approximation to the measure of ownership rights proposed here is used in Bebchuk et al. (2000).]

Both the measure for ownership and for control proposed here are then applied to one of the—if not the—largest case(s) of cross-holdings in the world, the conglomerate around Allianz and Münchener Rückversicherung in Germany, as it stood by the end of 2000. Using German data (Commerzbank 2001) on 11,181 firms and their respective owners, we compute the implied dividend rights and the control assignments that are compatible with the data. It turns out that the Allianz conglomerate is not only large itself, but also appears to be connected to the energy giant E.ON Aktiengesellschaft. The resulting conglomerate then accounts for a non-negligible part of the German economy.

According to the control assignments that we compute the 'Deutschland AG' may either be management controlled, foreign owned, controlled by public entities, or by holding companies, or it may be under the control of a single pool of investors, who also controls the German energy sector. As a matter of fact, the latter possibility turns out to have the strongest stability properties from a theoretical point of view, suggesting it as the most plausible control assignment.

The analysis, of course rests on assumptions. In particular, we treat ownership data as if it came from a pure one-share-one-vote world in which the simple majority rule obtains for shareholder assemblies. In practise there are many other devices to maintain control.

In the US, and to a certain extent in the U.K., regulation for minority shareholder protection and takeover bids, company bylaws with super-majority rules, and blockholder liability shield management from effective control by large shareholders (see Becht and Mayer 2001). In continental Europe (see Faccio et al. 2001), and to an even larger degree in East Asia (see Claessens et al. 2000), means such as voting caps, dual-class shares, voting trusts, and in particular cross-ownership among firms are used to achieve a separation of cash flow rights and voting rights (see also Becht and Mayer 2001).

This paper concentrates on the effects of cross-ownership and assumes the absence of other control arrangements.[1] But we expect the bias arising from this assumption to be small, because these other control devices will most likely reinforce the control assignment emerging from the pure one-share-one-vote arrangement, as they are presumably designed by controlling shareholders.

Otherwise the proposed measurements of both dividend rights and voting rights are derived purely from accounting identities, taking as given the existing ownership relations in an economy. They, therefore, constitute a benchmark computation for an ideal one-share-one-vote world without frictions. In particular, the present approach is preference-free in the sense that no information on the likelihood of coalitions among investors is used.[2]

The rest of the paper is organized as follows. Section 2 introduces the accounting identities and the notion of 'imputed shares' that assign dividend rights for arbitrary structures of cross-ownership. It also makes the notion of a pyramid precise, and illustrates the possibility of multiple control assignments. Section 3 presents the 'dominant shareholder theorem' on which the present notion of corporate control is based. In Sect. 4, the technique for calculating control coefficients is introduced formally and the concept of control assignments is defined. Section 5 describes the empirical results for the Allianz case. Section 6 concludes and provides a brief outline of further research.

2 Accounting

2.1 Imputed shares

Consider an economy with consumers/investors $i = 1, \ldots, n$ and firms $j = 1, \ldots, m$. Let $\xi_{ij} \geq 0$ denote the share of firm j owned privately by investor i, and denote by $\sigma_{ij} \geq 0$ the share in firm j owned by firm i (i.e., the first subscript denotes the owner and the second the owned firm). Denote the $n \times m$ matrix of privately held firm shares by $\Xi = \left[\xi_{ij}\right]$ (where $i = 1, \ldots, n$ and $j = 1, \ldots, m$) and the $m \times m$ matrix of firm shares held by other firms by $\Sigma = \left[\sigma_{ij}\right]$ (where $i, j = 1, \ldots, m$). These two matrices form the primitive data for the concepts to be developed.

By definition, for each firm j it must be true that $\sum_{i=1}^{m} \sigma_{ij} + \sum_{i=1}^{n} \xi_{ij} = 1$. Writing $e = (1 \ldots 1)$ for the summation (row) vector, in matrix notation this boils down to

$$e\Sigma + e\,\Xi = e\,. \tag{1}$$

[1] Dual-class shares can be taken into account by suitably renormalizing shares. Voting caps can similarly be accounted for by applying a minimum function (that takes the smaller of the cap and the controlled votes) to controlled shares.

[2] This distinguishes the present approach from one that relies, for instance, on the Shapley value. The latter implicitly assumes that all coalitions are equally likely—an assumption that appears unlikely in reality.

Assume that there is an upper limit δ on the share of a firm that can be held by other firms, with $0 < \delta < 1$. This condition is sufficient (but not necessary) for the matrix $(I - \Sigma)$ (where I denotes the identity) to possess an inverse. For, then $e\Sigma \leq \delta e \ll e$ implies that the matrix $(I - \Sigma)$ has a dominant diagonal, because $1 - \sigma_{jj} > \sum_{i \neq j} |-\sigma_{ij}|$ for all firms j.[3] Hence, $(I - \Sigma)^{-1}$ exists and (1) can equivalently be written as $e\Xi (I - \Sigma)^{-1} = e$. The $n \times m$ matrix Θ of *imputed ownership shares* θ_{ij} of investors $i = 1, \ldots, n$ in firms $j = 1, \ldots, m$ is, therefore, given by

$$\Theta = [\theta_{ij}] = \Xi \ (I - \Sigma)^{-1} \ . \tag{2}$$

In other words, if investor i holds *direct* shares $\xi_i = (\xi_{ij})_{j=1}^m$ in firms, his ultimate shares in the (net present value of) dividends ('imputed shares') are given by (the row vector)

$$\theta_i = (\theta_{ij})_{j=1}^m = \xi_i \ (I - \Sigma)^{-1} \tag{3}$$

for all $i = 1, \ldots, n$. [The same computation appears in Bolle and Güth (1992), Spagnolo (1998), and Gilo et al. (2006). Since $(I - \Sigma)^{-1} = \sum_{t=0}^{\infty} \Sigma^t \approx I + \Sigma$ the ownership measure in Bebchuk et al. (2000) can be viewed as a first-order approximation to (2).]

Let $\pi = (\pi_j)_{j=1}^m$ denote the (column) vector of firms' *cash flows* (or net present value of dividends, or liquidation values) and $\Pi = (\Pi_j)_{j=1}^m$ the (column) vector of *book values* (or profits) for firms. The book value Π_j of firm j consists of its cash flow plus the revenue from ownership in other firms, i.e. $\Pi_j = \pi_j + \sum_{h=1}^m \sigma_{jh} \Pi_h$, for all $j = 1, \ldots, m$. In matrix notation this boils down to

$$\Pi = \pi + \Sigma \Pi = (I - \Sigma)^{-1} \pi \ . \tag{4}$$

In other words, the sum of investor i's direct shares in book values equals the sum of his imputed shares in cash flows,

$$\xi_i \Pi = \xi_i \ (I - \Sigma)^{-1} \pi = \theta_i \pi$$

for all $i = 1, \ldots, n$.[4] In the aggregate, then, the sum of all book values equals the sum of cash flows, i.e.

$$e \ \Xi \ \Pi = e \ \Xi \ (I - \Sigma)^{-1} \pi = e\Theta\pi = e\pi \ , \tag{5}$$

using (1). We illustrate the concept of imputed shares for a version of the fictitious 'Noren Inc.' example from Sect. 1.

[3] For vectors, standard notation for inequalities is used: $x \geq y$ means that every coordinate of x is at least as large as the corresponding coordinate of y, $x > y$ means that $x \geq y$ and at least one coordinate of x exceeds the corresponding coordinate of y, and $x \gg y$ means that all coordinates of x exceed the corresponding coordinates of y.

[4] Of course, the same calculation applies if investors partake in firms' output vectors rather than profits.

Example 1 (Noren Inc.) Let there be two investors $i = 1,2$ and three firms $j = 1,2,3$ with

$$\Sigma = \begin{pmatrix} 0 & 0 & 0 \\ 1/4 & 0 & 1/2 \\ 1/4 & 1/2 & 0 \end{pmatrix} \quad \text{and} \quad \Xi = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon \\ 1/2-\varepsilon & 1/2-\varepsilon & 1/2-\varepsilon \end{pmatrix}$$

for some small $\varepsilon \geq 0$. Obviously, investor $i = 1$ stands for Mr. Len Kay and investor $i = 2$ represents "the public." Using (2), the imputed shares are given by

$$\Theta = \begin{pmatrix} 2\varepsilon & 2\varepsilon & 2\varepsilon \\ 1-2\varepsilon & 1-2\varepsilon & 1-2\varepsilon \end{pmatrix}\Bigg|_{\varepsilon=0} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

i.e., investor $i = 2$ almost fully owns the dividend rights in all three firms.

2.2 Controlled shares

As argued in Sect. 1, complicated ownership structures may divorce stock ownership from control over a company at almost no cost. The most obvious example of this is a "pyramid" or, more appropriately, a "chain."

Example 2 (Pyramid) Let Σ be given by $0 < \sigma_{ij} < 1$ for $j = i-1 \geq 1$ and $\sigma_{ij} = 0$ otherwise, i.e. for $i \geq 2$ company i owns $\sigma_{i,i-1} > 0$ shares in company $i-1$, but in no other company, and company $i = 1$ owns no firm shares at all. The inverse matrix required to compute imputed shares is easily identified as $A = \left[\alpha_{ij}\right] = (I-\Sigma)^{-1}$, where the entries α_{ij} are given by

$$\alpha_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ \prod_{h=j}^{i-1} \sigma_{h+1,h} & \text{if } 1 \leq j \leq i-1 \\ 0 & \text{if} \quad i < j \end{cases}$$

For, then $\alpha_{ij} - \sigma_{i,i-1}\alpha_{i-1,j} = 0$ if $i < j$, $\alpha_{ii} - \sigma_{i,i-1}\alpha_{i-1,i} = 1$ (for diagonal elements), and $\alpha_{ij} - \sigma_{i,i-1}\alpha_{i-1,j} = \prod_{k=j}^{i-1} \sigma_{h+1,h} - \sigma_{i,i-1} \prod_{k=j}^{i-2} \sigma_{h+1,h} = 0$ if $i > j$.

Assume for simplicity that all cross-holdings are of the same magnitude, i.e. $\sigma_{j,j-1} = \sigma > 0$, and that only the target firm $j = 1$ makes positive profits, while all other firms only live on their shareholdings, i.e. $\pi_1 > 0 = \pi_j$ for all $j \geq 2$. Then the entries of the matrix $A = (I-\Sigma)^{-1}$ become $\alpha_{ij} = \sigma^{i-j}$ if $1 \leq j \leq i$ and $\alpha_{ij} = 0$ if $i < j$. Hence, book values are given by $\Pi_j = \sigma^{j-1}\pi_1$ and imputed shares by $\theta_{kj} = \sum_i \alpha_{ij}\xi_{ki} = \sum_{i \geq j} \sigma^{i-j}\xi_{ki}$ for any shareholder k and all firms j. Now consider a shareholder k who owns $\xi_{km} > 1/2$ shares of company $j = m$ and suppose that cross-holdings exceed 50%, i.e. $\sigma > 1/2$.

Then each company $j \geq 2$ holds a majority in company $j-1$ and, therefore, shareholder k has full control over all companies $j = 1, \ldots, m$. Conversely, if shareholder k controls firm $j = 1$, then she must also control firm $j = 2$, because otherwise the investor in control of firm 2 will hold more shares in company 1. Since the same argument applies to any firm j with $1 < j < m$, control of firm 1 implies control of all firms. Hence, an investor controls firm $j = 1$ *if and only* if she controls all firms in the chain. Yet the cost of buying majority control in company $j = 1$ by buying $\xi_{km} > 1/2$ amounts only to $\xi_{km}\sigma^{m-1}\pi_1 \approx 2^{-m}\pi_1 \to_{m\to\infty} 0$ if ξ_{km} and σ are close enough to 1/2.

For the example of a pyramid we now illustrate the key technique of computing control coefficients. (The formal development is deferred to Sect. 4.) Let $C = \left[c_{ij}\right]$ be an $n \times m$ matrix, the entries of which are zeros or 1's such that $eC = e$, i.e., for each firm j there is precisely one investor i such that $c_{ij} = 1$ (and $c_{hj} = 0$ for all $h = 1, \ldots, n$ with $h \neq i$). The coefficients c_{ij} are chosen such that $c_{ij} > 0$ *if and only if* investor i is in control of firm j. In Example 2 this yields $c_{km} = 1$ from the assumption that $\xi_{km} > 1/2$. Thus, the shares that firm m holds in other firms are now reassigned to investor k, so that k controls $\xi_{k,m-1} + c_{km}\sigma_{m,m-1} \geq \sigma_{m,m-1} = \sigma > 1/2$ shares in firm $m-1$. Hence, investor k is also in control of firm $m-1$, implying that $c_{k,m-1} = 1$. Therefore, k controls $\xi_{k,m-2} + c_{k,m-1}\sigma_{m-1,m-2} \geq \sigma_{m-1,m-2} = \sigma > 1/2$ shares in firm $m-2$, and so on. Ultimately, the matrix C contains all zeros except for the k-th row which consists of 1's, expressing that investor k controls all firms in the pyramid. The controlled shares are now given by $\Xi + C\Sigma$, i.e., all firm shares held by other firms are reassigned to investor k in this example.

The distinguishing feature of the pyramid is that there is no other way to assign control coefficients, other than $c_{kj} = 1$ for all $j = 1, \ldots, m$. This may not be the case for different structures of cross-ownership. The simplest example for this is a "ring" that involves only two firms.

Example 3 Suppose there are two firms $j = 1, 2$ and two investors $i = 1, 2$, and assume that $\sigma_{3-j,j} > 1/2 - \min_{i=1,2}\xi_{ij}$ (and $\sigma_{jj} = 0$) for $j = 1, 2$. Then, for each $i = 1, 2$, assigning control coefficients $c_{ij} = 1$ (and $c_{3-i,j} = 0$) for both $j = 1$ and $j = 2$ indeed identifies investor i as the controlling shareholder for both companies, independently of how small ξ_{ij} is. For, with this assignment $\xi_{ij} + c_{i,3-j}\sigma_{3-j,j} \geq \min_{k=1,2}\xi_{kj} + \sigma_{3-j,j} > 1/2$ shows that if i controls both companies, he commands an absolute majority of votes in both firms. Hence, the assignment of control coefficients is consistent with majority voting. Since this holds for both $i = 1$ and for $i = 2$, we speak of a *multiplicity* of control assignments in such a case.

Apart from the two identified above, there are *no other* assignments of control coefficients that are consistent. For, suppose that $c_{ij} = 1$ (and $c_{3-i,j} = 0$) and $c_{3-i,3-j} = 1$ (and $c_{i,3-j} = 0$) for $i, j = 1, 2$. Then it follows that, for $j = 1, 2$,

$$\xi_{ij} + c_{i,3-j}\sigma_{3-j,j} = \xi_{ij} < \xi_{3-i,j} + c_{3-i,3-j}\sigma_{3-j,j} = \xi_{3-i,j} + \sigma_{3-j,j},$$

because $\xi_{ij} = 1 - \xi_{3-i,j} - \sigma_{3-j,j}$ and $\sigma_{3-j,j} + \xi_{3-i,j} \geq \sigma_{3-j,j} + \min_{k=1,2}\xi_{kj} > 1/2$ by assumption. Therefore, $3-i$ controls more votes than i, in contradiction to $c_{ij} = 1$. Since this holds for both $i = 1$ and $i = 2$, it implies that the *only* assignments of control coefficients, that are consistent with majority voting, are those where *one* investor controls *both* firms.

As a particular numerical instance let the share distribution be given by

$$\Sigma = \begin{pmatrix} 0 & 0.2 \\ 0.2 & 0 \end{pmatrix} \quad \text{and} \quad \Xi = \begin{pmatrix} 0.47 & 0.33 \\ 0.33 & 0.47 \end{pmatrix},$$
$$\text{such that } \Theta = \begin{pmatrix} 0.56 & 0.44 \\ 0.44 & 0.56 \end{pmatrix}.$$

Then each firm $j = 1,2$ has clearly a majority owner of its (imputed) dividend rights, as $\theta_{ii} = 0.56 > 1/2$ for $i = 1,2$. Still, the majority owners of dividend rights cannot both control "their" firms. This is, because control can only be assigned such that one investor controls both firms.

The difference between the "ring" in Example 3 and the "pyramid" of Example 2 is as follows. The ring minimizes the number of firms involved in maintaining control. But, if a change in management for one or both firms occurs, the control structure may change. (There are multiple "control assignments.") The pyramid, on the other hand, guarantees control, but involves a large number of firms, at least if a small cost of maintaining control is desired.

To assess the incentives for and the consequences of such constructions would require specifying a game, where investors strategically attempt to gain control by buying into or founding e.g. holding companies or cartels which own each other. Though such a model would be of imminent interest, it is beyond the scope of the present paper.

This paper is confined to a tool that allows one to determine whether—for a *given* structure of cross-ownership and the simple majority rule—a particular control assignment may or may not obtain. For example, it is clear that the 'Len Kay' from Sect. 1 (Example 1) can exercise full control while holding virtually none of the dividend rights of the target firm. But, as 'Noren Inc.' has a ring structure, there is an alternative 'control assignment' in which the majority holder of dividend rights (with respect to imputed shares; see Example 1) exercises control over all firms.

These examples are straightforward, because they involve only two investors and few firms. With more than two investors there may be no shareholder, who alone controls an absolute majority of the voting shares. Such cases then require a more general notion of when an investor is "in control" of a company.

3 Shareholder assembly

When can an investor i exercise control over a company j? To develop intuition, consider a particular firm j and assume for the moment that no other firm holds shares in firm j, i.e. $\sigma_{hj} = 0$ for all $h = 1, \ldots, m$. Then, under a simple majority rule for shareholder assemblies, investor i will certainly be able to control company j if he privately owns at least 50% of all shares in firm j, i.e. if $\xi_{ij} \geq 1/2$. Hence, at least 50% privately held shares are sufficient for control.

Owning an absolute majority of shares, however, may not be necessary. Once a shareholder "controls" a company it takes voting in a shareholder assembly to unseat him. Think of voting in a shareholder assembly as a "subgame" that is reached when shareholders are dissatisfied with management's decisions. If there is an equilibrium in the "voting subgame" that supports the status quo, then the controlling shareholder will not be unseated by such a challenge and management can run the company at will.

The bite of the following 'dominant shareholder theorem' is that it *characterizes* the (pure strategy) equilibria of the "voting subgame." As long as management's decision—the status quo—is compatible with the will of the largest shareholder, it cannot be defeated with certainty in a shareholder assembly. Therefore, if management strives to avoid being overthrown by an assembly, it can do so by executing the preferred decisions of the largest shareholder.

We now describe the "voting subgame" in more detail. Suppose that in company j two alternatives—the status quo and an alternative proposal—have to be voted on in a shareholder assembly. Every shareholder of company j has to decide whether or not to participate in the assembly and, if he decides to do so, how to cast his votes. Assume that participation in the assembly carries a small, but positive and privately born *cost*. Also assume that the shareholder assembly decides with *simple* majority of represented votes, and that *ties* are broken in favor of the status quo.

Denote by A the set of shareholders, who (strictly) prefer the status quo, and by B the set of shareholders, who (strictly) support the alternative proposal. Shareholders, who are indifferent, will not participate in the assembly in order to avoid the cost of participation. Therefore, it can be assumed without loss of generality that $A \cup B = \{1, \ldots, n\}$, where n is the number of shareholders.

This defines a game among shareholders. Preferences of the players are as follows: for shareholder $i \in A$ (resp. $i \in B$) the best outcome is the status quo (resp. the alternative) and not participating, the second-best is the status quo (resp. the alternative) and participation, which is better than the alternative (resp. the status quo) without participation, and the worst is the alternative (resp. the status quo) with participation. Strategies are either to participate and vote for the preferred outcome, or not to participate. (With stochastic participation costs, participating and voting against one's preference is a strictly dominated strategy and, therefore, not considered here.)

The following 'dominant shareholder theorem' is proven in Ritzberger (2005) for the game just defined: there exists a pure strategy Nash equilibrium if and only if there is a shareholder $k \in B$, who opposes the status quo, and holds a larger share of the votes than any one of the supporters of the status quo, i.e. $\xi_{kj} > \xi_{ij}$ for all $i \in A$. In this equilibrium the status quo is defeated with certainty by shareholder k voting against it.

Since this is a characterization of pure strategy equilibria, it pins down a management policy which is never defeated with certainty—whatever the alternative and shareholder preferences are: if management executes the will of (one of) the largest shareholder(s), then the characterizing condition must fail. For, then there is always a supporter $i \in A$ of the status quo (the relative majority owner, who sets the status quo) such that $\xi_{ij} \geq \xi_{kj}$ for all challenging shareholders $k \in B$, so that there is *no* pure strategy equilibrium in the "voting subgame," where the status quo gets dismissed. Therefore, if management strives to survive the threat of shareholder assemblies, it ought to behave according to the preferences of (one of) the largest shareholder(s). If it violates the interests of all relative majority holders, then there is an equilibrium in the voting subgame, where it is overthrown with certainty. This makes relative majority holders "controlling" shareholders.

This argument has two drawbacks. First, if management behaves according to the preference of a relative majority owner, then there is still a (nonzero) chance that a shareholder assembly will vote against it. This is because, under this premise, there are only *mixed* equilibria in the voting subgame. Second, the argument relies on the status quo being adopted as the default, even if no shareholder participates in the assembly at all. Yet, the tie-breaking in favor of the status quo is most plausible, when some supporter of the status quo holds the tie-breaking vote that he can exercise at his discretion. Under most corporate charters this would apply to the president of the company. But the latter requires that the president is committed to participate. If this is the case, the set of pure equilibria changes.

Therefore, consider now a modified voting game, where some supporter $\iota \in A$ of the status quo—the president, say—is committed to participate (and vote for the status quo). Otherwise, the assumptions are as before. Let a_i denote the probability that shareholder i will participate and cast his votes in accordance with his preference. Then, the set of pure strategy Nash equilibria is characterized as follows.

Proposition 4 *The pure strategy combination $a = (a_1, \ldots, a_n) \in \{0,1\}^n$ with $a_\iota = 1$ (where $\iota \in A$ is the committed shareholder) is a Nash equilibrium* if and only if *either*

(a) *the committed supporter ι of the status quo holds a share at least as large as any one of the opponents of the status quo, i.e. $\xi_{\iota j} \geq \xi_{ij}$ for all $i \in B$, and only ι participates, i.e. $a_i = 0$ for all $i = 1, \ldots, n$ with $i \neq \iota$, in which case the status quo is adopted with certainty; or*

(b) *there is a subset $B^* \subseteq B$ of opponents of the status quo such that this subset jointly holds more votes than the committed supporter ι and the largest supporter of the status quo other than ι together do, but such that if any one of the opponents in B^* would withdraw, the remaining coalition would not hold more votes than ι, i.e.*

$$\begin{gathered} \xi_{\iota j} \geq \textstyle\sum_{k \in B^* \setminus \{i\}} \xi_{kj} \quad \textit{for all } i \in B^* \\ \textit{and } \textstyle\sum_{k \in B^*} \xi_{kj} > \xi_{\iota j} + \max_{k \in A \setminus \{\iota\}} \xi_{kj} \end{gathered} \tag{6}$$

and only ι and the members of B^* participate, i.e. $a_i = 1$ for all $i \in B^*$ and $a_i = 0$ for all $i \in A \setminus \{\iota\}$ and all $i \in B \setminus B^*$, in which case the status quo is dismissed with certainty.

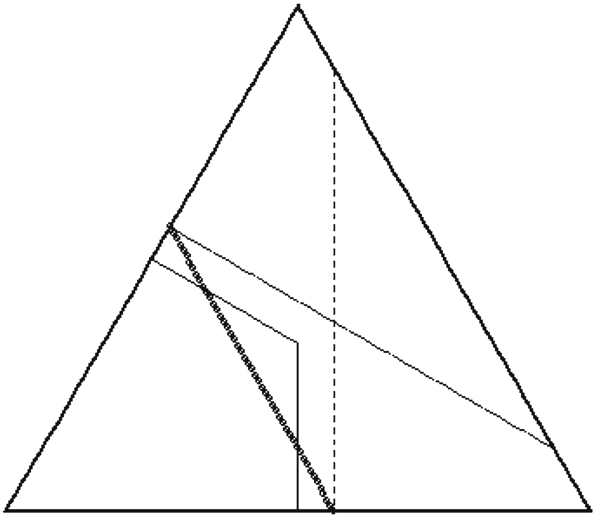

Fig. 1 Regions of pure strategy equilibria for $\xi_{1j} = 1/9$

Proof See Appendix.

To see the contents of this proposition, consider an example with four shareholders, where $\iota = 2 \in A = \{1, 2\}$ is the (committed) "president," $i = 1$ is a supporter of the status quo holding a share of $\xi_1 = 1/9$, and investors $i = 3$ and $i = 4$ oppose the status quo, i.e. $B = \{3, 4\}$. The proposition describes two (exhaustive) possibilities for the existence of a pure Nash equilibrium. If, as in part (a), the committed investor $\iota = 2$ holds at least as many shares as either 3 or 4 do individually, then there is a Nash equilibrium in which the status quo is adopted with certainty. If, as in part (b), there is a subset B^* of $B = \{3, 4\}$ that holds collectively more votes than the supporters $\iota = 2$ and $i = 1$ do together, and every member of B^* is "pivotal," then there is another Nash equilibrium in which the status quo is dismissed with certainty. In the example B^* could be the 'grand' coalition of $i = 3$ and $i = 4$ together, i.e. $B^* = B = \{3, 4\}$, or a singleton, i.e. $B^* = \{3\}$ or $B^* = \{4\}$. For the 'grand' coalition $B^* = \{3, 4\}$ condition (6) would amount to $\xi_{2j} \geq \xi_{3j}$, $\xi_{2j} \geq \xi_{4j}$, and $\xi_{3j} + \xi_{4j} > \xi_{1j} + \xi_{2j}$, i.e., none of the two contenders 3 and 4 has more votes that the "president," but jointly they control more votes than the "president" $\iota = 2$ and his supporter 1 together do. For a singleton, say, $B^* = \{3\}$, the first part of condition (6) is irrelevant and the second states that 3 controls more votes than $\iota = 2$ and 1 together do, i.e. $\xi_{3j} > \xi_{1j} + \xi_{2j}$.

The geometry of this example is visualized in Fig. 1. The bottom left vertex corresponds to $\xi_{2j} = 8/9$, the bottom right vertex to $\xi_{3j} = 8/9$, and the top vertex to $\xi_{4j} = 8/9$. South-west of the two thin solid lines Proposition 4(a) holds true. The little triangle that is cut out by the steep dotted line from the region, where Proposition 4(a) obtains, is the region, where the status quo can successfully be challenged by the 'grand' coalition $B^* = B = \{3, 4\}$. This is, therefore, a region where two types of equilibria co-exist, one where the status quo is adopted—as in Proposition 4(a)—and one where it is dismissed—as in Proposition 4(b); see the Remark below. South-west of the thin solid lines *and* south-west of the steep dotted line is where only the equilibrium described in Proposition 4(a) obtains, i.e. where the status quo survives for sure.

To the east (right) of the vertical broken line is the region, where $B^* = \{3\}$ can topple the status quo; and to the north-east of the downward sloping broken line $B^* = \{4\}$ can topple the status quo. The union of the region to the east of the vertical broken line with the region to the north-east of the downward sloping broken line is where the status quo is always dismissed in equilibrium, i.e. where only Proposition 4(b) obtains. There remain the 'kinked road' north-east of the thin solid lines and south-west of the two broken lines, where no (pure strategy) equilibrium exists. There, both 3 and 4 command more votes individually than $\iota = 2$, but not more than $\iota = 2$ together with $1 \in A$. If one of them would participate alone, 1 would become pivotal and, therefore, also participate. But if 1 does so, 3 and 4 have to participate simultaneously in order to win the vote. But if they both participate, 1 is no longer pivotal.

That the 'kinked road,' where no pure equilibria exist, lives on the terrain, where either 3 or 4 own a relative majority of votes, captures "persistence" of control. Once $\iota = 2 \in A$ is in control, it is not enough for a raider $i = 3, 4$ to purchase more shares than $\iota = 2$ holds; in order to take over with certainty, i must buy more shares than what $\iota = 2$ and the largest supporter of the status quo other than $\iota = 2$ (that is, 1) hold together.

The latter consequence of Proposition 4 makes it particularly useful for the present purpose. If a controlling shareholder (relative majority owner) can make sure that sufficiently many outstanding shares are in the hands of "friendly" entities—like other companies affiliated through cross-ownership—these shares will form a barrier against take-over attempts, even when his own holdings fall short of an absolute majority. These "friendly" entities will *not* vote in equilibrium at shareholder assemblies, but their mere existence will raise the hurdle for a unilateral take-over attempt [they will hold the shares $\max_{k \in A \setminus \{\iota\}} \xi_{kj}$ from (6)]. This observation explains why the "ring" construction from Example 3 works in the interest of the controlling shareholder.

A key advantage of the version of the 'dominant shareholder theorem' in Proposition 4 is that, once a company is controlled by (one of) its largest shareholder(s), there is actually a pure strategy equilibrium in the voting subgame, where the status quo does survive with certainty. The following pins down when equilibrium outcomes are unique.

Corollary 5 (a) *If there is an opponent $i \in B$ of the status quo, who controls more votes than the "president" $\iota \in A$ together with the largest supporter of the status quo other than ι, i.e. such that $\xi_{ij} > \xi_{\iota j} + \max_{k \in A \setminus \{\iota\}} \xi_{kj}$, then there exists a pure strategy equilibrium and in all pure strategy equilibria the status quo is dismissed.*

(b) *If the "president" $\iota \in A$ controls at least as many votes as any one of the opponents of the status quo, i.e. $\xi_{\iota j} \geq \xi_{ij}$ for all $i \in B$, and the "president" ι together* *with the largest supporter of the status quo other than ι control at least as many votes as all opponents of the status quo do together, i.e.* $\xi_{\iota j}+\max_{k\in A\setminus\{\iota\}}\xi_{kj} \geq \sum_{i\in B}\xi_{ij}$, *then there exists a pure strategy equilibrium and in all pure strategy equilibria the status quo is adopted.*

Proof See Appendix.

Remark 6 If the status quo can successfully be challenged by the grand coalition $B^* = B$ of all opponents, then it may as well survive. That is, condition (6) from Proposition 4(b) can only be fulfilled for $B^* = B$, if condition (a) from Proposition 4 holds at the same time. For, suppose that (6) holds for $B^* = B$. Then it implies

$$\xi_{\iota j} \geq \sum_{k\in B\setminus\{h\}} \xi_{kj} \geq \xi_i \quad \text{for all } i, \quad h \in B \quad \text{with } i \neq h,$$

so $\xi_{\iota j} \geq \xi_{ij}$ for all $i \in B$, viz. condition (a) from Proposition 4. Therefore, if condition (b) holds for $B^* = B$, then there are multiple pure equilibria. For those pure equilibria, where the status quo gets dismissed, the coordination requirements among shareholders in B are demanding, however.

All this, of course, obtains when preferences of shareholders are given and voting is on a fixed alternative versus the status quo. If shareholder preferences or the alternative vary, the sets of supporting and challenging shareholders, A and B, vary as well. In the worst case the set A of supporters of the status quo becomes empty and management is defeated in a shareholder assembly with certainty. On the other hand, the status quo is chosen by management. And Proposition 4 shows that there is a policy which is always supported by a pure equilibrium in the voting subgame, whatever the alternative and shareholder preferences are: management executes the will of (one of) the largest shareholder(s).

In this sense a *relative* majority of voting shares is not only necessary, but also sufficient for control. This is the first reason to take relative majority as the criterion for control in the sequel. (A second reason is explained in Example 7 below.) The only modification that arises in the presence of cross-ownership is that the relative majority may be owned not only directly, but also indirectly.

4 Control assignments

4.1 Control coefficients

The above argument about control does not account for cross-ownership among firms. In the presence of the latter privately held shares ξ_{ij} are not the only ones that an investor i can use to control a company j. If he controls enough other companies (other than j), such that their combined shares in firm j (together with his privately held shares) make for a majority, then i will still be able to exercise control over firm j. (Recall 'Noren Inc.' and Example 3.) Therefore, under cross-ownership, finding possible control assignments requires *simultaneous* consideration of all firms and investors. That is, the privately held shares ξ_{ij} need to be replaced by the sum of privately held shares plus the shares commanded by any other firm that is controlled by the investor under scrutiny either directly or indirectly.

This can be achieved by specifying an equation system, the solutions of which are *control coefficients* c_{ij} for all $i = 1, \ldots, n$ and all $j = 1, \ldots, m$. We think of the latter as an indicator function that takes a positive value precisely if investor i can control company j (and is zero otherwise). If i is unrivaled in controlling firm j, the control coefficient c_{ij} takes the value 1, in order to assign the votes associated with the shareholdings of company j in other firms to the controlling shareholder i. For all investors other than i the control coefficients (for firm j) are then zero, expressing that i has full control over company j. If i has a rival $h \neq i$ who is equally powerful in his shareholdings in firm j (e.g. both holding 50%), then control coefficients may split control such that $c_{ij} > 0$, $c_{hj} > 0$, and $c_{ij} + c_{hj} \leq 1$. If i and h are, moreover, the only individuals who can control company j, then we would want $c_{ij} + c_{hj} = 1$.

More concretely, let $C = \left[c_{ij}\right]$ be such a nonnegative matrix of control coefficients, where $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Then the (shares of) votes in firm j controlled by individual i are given by $\xi_{ij} + \sum_{k=1}^{m} c_{ik}\sigma_{kj}$. After this reassignment of shares, the voting shares that remain uncontrolled within the firm sector are reduced to $(I - \operatorname{diag}(eC))\,\Sigma$, where $\operatorname{diag}(eC)$ denotes the $m \times m$ diagonal matrix with the column sums of C on its main diagonal. Because of (1), it follows that

$$e\Xi + eC\Sigma + e\,(I - \operatorname{diag}(eC))\,\Sigma = e\,,$$

so that the modified shareholding matrices, $\Xi + C\Sigma$ and $(I - \operatorname{diag}(eC))\,\Sigma$, represent a valid share distribution. These modified shares differ from the imputed shares in (2), because control over a firm implies control over all the voting shares in other firms that this firm holds.

But, unless the matrix $(I - \operatorname{diag}(eC))$ is identically zero, some firm-held shares remain uncontrolled by investors. Therefore, the result of such a *partial* reassignment of voting rights can be drastically at variance with what would result if all voting rights were assigned to investors. The following example illustrates this point. There we adopt, for a moment, *absolute* majority as the criterion relevant for control. In the example this implies that some firm-held shares remain "uncontrolled." As a consequence, a completely different control assignment emerges as compared to the case, where the control-criterion of *relative* majority ensures that all control rights are held by investors.

Example 7 Let the matrices Ξ of privately held shares and Σ of firm held shares be given by

$$\Xi = \begin{pmatrix} 1/5 & 1/5 & 5/16 \\ 3/20 & 3/20 & 3/8 \\ 3/20 & 3/20 & 5/16 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0 & 1/10 & 0 \\ 1/10 & 0 & 0 \\ 2/5 & 2/5 & 0 \end{pmatrix}.$$

If any firm is controlled by a shareholder with absolute majority (of directly and indirectly held shares), then investor $i = 1$ will control firms $j = 1$ and $j = 2$: first, firm $j = 3$ cannot be controlled with absolute majority, because no other firm holds shares in $j = 3$ and no investor holds privately an absolute majority in firm $j = 3$. Therefore, control coefficients for firm $j = 3$ for the absolute majority criterion must be zero for all $i = 1, 2, 3$. But then neither investor 2 nor investor 3 can control more than 3/20 of the votes in firm 1 or firm 2; and 3/20 of all votes would not even survive a challenge by investor $i = 1$ alone, who privately controls $1/5 = 4/20$ of the votes in firms 1 and 2. Hence, control coefficients for firms 1 and 2 for the absolute majority criterion must also be zero for investors 2 and 3. On the other hand, given that $i = 1$ controls firms 1 and 2, he controls 1/5 (his privately held share) plus 1/10 (the share of firm $3 - j$), i.e. a total of 3/10, of all shares in firm j, for $j = 1, 2$. This is enough to block any challenge from investors 2 and 3, who jointly also control only 3/10 of the vote in companies $j = 1, 2$; hence, investor 1's control coefficients for firms 1 and 2 can be positive under the absolute majority criterion.

But with *relative* majority investor $i = 2$ must be able to control firm $j = 3$, because he holds a relative majority (3/8) privately. Investor 2's control coefficient for firm 3 for the relative majority criterion is 1, because there is no other largest shareholder. It follows that $i = 2$ controls at least 3/20 (his privately held shares) plus 2/5 (the shares held by firm 3), i.e. a total of 11/20, of all shares in firms $j = 1, 2$. Since no other investor can combine more than $1/5 + 1/10 = 3/10$ of all shares in firms $j = 1, 2$, investor 2 controls all firms with relative majority, while investor 1 controls none of the firms. (Indeed, investor 2 controls an absolute majority in firms 1 and 2, as $11/20 > 1/2 > 7/20 = \xi_{1j} + \xi_{3j}$, for $j = 1, 2$.)

The reason for this radical change in the control assignment when passing from absolute to relative majority is that at the absolute majority level firm 3 remains "uncontrolled." Its shares of 2/5 in firms 1 and 2 are left inactive, as if it were committed to abstain from any voting at all. But suppose that investors 2 and 3 team up to elect $i = 2$ as the CEO for company 3. Since they jointly command $5/16 + 3/8 = 11/16 > 1/2$ of all shares in firm 3, they can do so even at the absolute majority level. But this brings firm 3's shares of 2/5 in companies 1 and 2 under the control of investor 2, enabling him to take control in all firms.

This example shows that once company $j = 3$ is under control (of investor $i = 2$) with relative majority, the controlling investor can seize control of all other companies even under the criterion of absolute majority. Therefore, leaving some firm-held shares "uncontrolled" may give a biased picture. Adopting relative majority as the relevant criterion for control has the advantage, that all firm-held votes will be reassigned to investors, because a largest shareholder always exists. Thereby, relative majority avoids the bias introduced by "uncontrolled" shares. This constitutes our second reason, besides Proposition 4, to opt for relative majority as the relevant control-criterion.

Summarizing the findings, we seek to determine a nonnegative real $n \times m$ matrix $C = [c_{ij}]$ of control coefficients so as to satisfy the two following two conditions:

(1) If $c_{ij} > 0$, then $\xi_{ij} + \sum_{k=1}^{m} c_{ik}\sigma_{kj} \geq \xi_{hj} + \sum_{k=1}^{m} c_{hk}\sigma_{kj}$ for all $h = 1, \ldots, m$, for all $j = 1, \ldots, m$ and all $i = 1, \ldots, n$;
(2) $\sum_{i=1}^{n} c_{ij} = 1$ for all $j = 1, \ldots, m$.

The first property says that only relative majority shareholders can control a company; the second that every company is controlled by some investor.

We turn next to the formalism required to determine control coefficients. First, considering every firm in isolation, let $\Delta = \left\{x \in \mathbf{R}_{+}^{n} \left| \sum_{i=1}^{n} x_i = 1\right.\right\}$ be the simplex of all distributions of shares in a given firm. For each $x \in \Delta$ let

$$\begin{aligned} W(x) &= \{i = 1, \ldots, n \,|\, x_i \geq x_k, \text{ for all } k = 1, \ldots, n\} \\ &= \arg \max_{i=1,\ldots,n} x_i \end{aligned} \tag{7}$$

be the set of maximal coordinates of x. For any subset $W \subseteq \{1, \ldots, n\}$ of investors denote by $\Delta(W) = \left\{x \in \Delta \left| \sum_{i \in W} x_i = 1\right.\right\}$ the 'face' of Δ spanned by W, i.e. the share distributions with support in the set of maximal coordinates of x. Using these two conventions, define the correspondence $f : \Delta \to \Delta$ by

$$f(x) = \Delta(W(x)) \tag{8}$$

This correspondence assigns to every vector $x \in \Delta$ the face spanned by the maximal coordinates of x. It satisfies that (1) it assigns a positive value only if the corresponding shareholder commands a relative majority of votes, i.e. for all $i = 1, \ldots, n$, if $f_i(x) > 0$ then $x_i \geq x_k$ for all $k = 1, \ldots, n$ and all $x \in \Delta$, and (2) it assigns all voting rights to (largest) shareholders, i.e. $ef(x) = 1$ for all $x \in \Delta$.

The construction of f is for a single firm. To capture the simultaneity among all firms, define the correspondence $F : \Delta^m \to \Delta^m$, for an $n \times m$ matrix $X = [x^1 \ldots x^m] \in \Delta^m$ with columns $x^j \in \Delta$ for all $j = 1, \ldots, m$, by

$$F(X) = [f(x^1) \ldots f(x^m)], \tag{9}$$

where f is given by (8). Note that $C \in F(X)$ implies $eC = e$ for any matrix $X \in \Delta^m$, because $f(\cdot) \subseteq \Delta$.

Definition 8 A *control assignment* is a nonnegative real $n \times m$ matrix $C = [c_{ij}]$ of control coefficients such that

$$C \in F(\Xi + C\Sigma). \tag{10}$$

According to this specification, the control coefficient c_{ij} is positive at a solution to (10) only if i controls at least as many votes (according to $\Xi + C\Sigma$) as any other investor. And, that F maps into Δ implies that $eC = e$ holds. In other words, solutions to (10) satisfy the two conditions (1) and (2) above.

Technically, it is clear from (8) that F as in (9) constitutes an upper hemi-continuous nonempty- and convex-valued correspondence from the compact and convex subset Δ^m of an Euclidean space into itself. Therefore, it meets the hypotheses of Kakutani's fixed point theorem and a solution to (10) exists.

Proposition 9 *For every pair of share matrices* Ξ *and* Σ *there exists a control assignment.*

A control assignment constitutes a consistent way to reassign firm-held shares to investors such that (1) an investor has control over a firm only if he controls a relative majority in this firm directly and indirectly, and (2) all firm-held shares are reassigned to investors. Given a control assignment C, the $n \times m$ matrix of voting shares controlled by investors in the various firms is then given by $\Xi + C\Sigma$.

Example 10 Returning to Example 7, it has been shown above that the matrix

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

constitutes a control assignment. The associated controlled voting shares are given by

$$\Xi + C\Sigma = \begin{pmatrix} 1/5 & 1/5 & 5/16 \\ 13/20 & 13/20 & 3/8 \\ 3/20 & 3/20 & 5/16 \end{pmatrix} = \begin{pmatrix} 0.2 & 0.2 & 0.3125 \\ 0.65 & 0.65 & 0.375 \\ 0.15 & 0.15 & 0.3125 \end{pmatrix}.$$

Thus, the controlling shareholder $i = 2$ ends up with 65% of the votes in firms 1 and 2, and with his relative majority of 37.5% in firm 3.

The notion of a control assignment also makes precise when a complete separation of ownership and control occurs. In particular, we may call a firm j *management controlled* if there exists a control assignment C such that $c_{ij} = 1$ for some i with $\xi_i = \left(\xi_{ij}\right)_{j=1}^m = 0$. That is, firm j is management controlled if it is controlled by an investor who holds no shares privately [and thus owns no dividend rights at all; see (3)].

4.2 How 'Tight' is control?

Relative majority, as captured by (10), is the property that the controlling shareholder commands at least as many votes as any other *single* shareholder. Intuitively, one may expect that his control is 'tighter' if he commands more votes than any coalition of other shareholders with *more than one* member.

An investor owns the largest block of votes, if he controls 34% of a company in which two other individuals own 33% each. His grip on the company appears 'tighter,' though, if he would own 19% and nine other shareholders would own only 9% each. For, then he would command more votes than any coalition of *two* other shareholders.

Since the ultimate decision units in voting are the shareholders, not the shares, the (maximum) size of challenging coalitions that a controlling shareholder can "block" provides a proxy for the power of the largest shareholder. Accordingly, to each control assignment C [that satisfies (10)] and each firm j an integer $\tau_j = 1, \ldots, n-1$ can be associated which measures the size of the largest coalition of shareholders (other than the largest owner) that owns no more votes than the controlling shareholder.

The intuition is as follows. Given a control assignment C, the controlled votes can be computed by $\Xi + C\Sigma$. For a firm j and a controlling shareholder i (i.e. such that $c_{ij} > 0$) one can then ask: What is the largest size of coalitions of shareholders other than i that do not control more votes than i does? For instance, in Example 10 investor 2 controls more votes than 1 and 3 *together* do in firms 1 and 2, but only more than what 1 or 3 control *individually* in firm 3. Accordingly, to the control assignment in Example 10 the vector $\tau = (2, 2, 1)$ would be assigned. This vector expresses that, under the control assignment C, in firms 1 and 2 the controlling shareholder commands at least as many votes as any coalition of *two* other shareholders, while in firm 3 the controlling shareholder holds at least as many votes as any *single* other shareholder (that is, if two other shareholders team up, they will have more votes).

Formally, let $\mathcal{N}_{it} = \{N \subset \{1, \ldots, n\} \mid i \notin N, \ |N| = t\}$ denote the set of all coalitions that do not contain i and have t members, for all $i = 1, \ldots, n$ and all $t = 1, \ldots, n-1$. For a share distribution $x \in \Delta$ let

$$W_t(x) = \left\{ i = 1, \ldots, n \,\middle|\, x_i \geq \sum_{h \in N} x_h \text{ for all } N \in \mathcal{N}_{it} \right\} \tag{11}$$

be the set of potential 'winners' and define $t(x)$ as the largest $t \in T$ such that $W_t(x)$ is nonempty. (Clearly, $W_1(x) = W(x) \neq \emptyset$, where the latter is defined in (7), but for $t > 1$ the set $W_t(x)$ may be empty.)

Remark 11 If $1 < t(x) < n-1$, then there is a unique $i = 1, \ldots, n$ such that $W_t(x) = \{i\}$. For, if $i \in W_t(x)$ at $t = t(x)$, then $x_i \geq \sum_{h \in N} x_h$ for all $N \in \mathcal{N}_{it}$ and, of course, $x_i < \max_{N \in \mathcal{N}_{is}} \sum_{h \in N} x_h$ for all $s > t$ by the definition of $t = t(x)$. It follows that $x_k \leq x_i$ for all $k = 1, \ldots, n$. If there were $k \neq i$ such that $x_k \geq \sum_{h \in N} x_h$ for all $N \in \mathcal{N}_{kt}$, then for any $M \in \mathcal{N}_{kt}$ with $i \in M$

$$x_i \geq x_k \geq \sum_{h \in M} x_h \geq x_i$$

would imply $x_i = x_k$ and $x_h = 0$ for all $h \neq i, k$. But then $\max_{N \in \mathcal{N}_{it}} \sum_{h \in N} x_h = x_k = \max_{N \in \mathcal{N}_{kt}} \sum_{h \in N} x_h = x_i$ and $x_h = 0$ for all $h \neq i, k$ would imply $t(x) = n-1$, in contradiction to the hypothesis. Therefore, i must be unique if $t(x) < n-1$. The same argument establishes that if $t(x) = n-1$, then $W_{n-1}(x)$ contains at most two elements. Only if $t(x) = 1$ the set $W_t(x) = W_1(x)$ can contain any number of elements between 1 and $n-1$.

Next, for an $n \times m$ matrix $X = [x^1 \ldots x^m]$ with columns $x^j \in \Delta$ for all $j = 1, \ldots, m$ define the vector $\tau(X) = [\tau_1(X), \ldots, \tau_m(X)]$ by $\tau(X) = [t(x^1), \ldots, t(x^m)]$.

To measure how tightly controlled firms are, the function τ gets applied to the matrix $\Xi + C\Sigma$ of controlled votes. For any control assignment C the associated "orders of control" τ_j $(j = 1, \ldots, m)$ will lie between 1, because a controlling shareholder must hold at least a relative majority, and $n-1$, in which case no coalition of others can unseat the controlling shareholder. In realistic cases the orders τ_j will lie between these bounds. If τ_j is, for instance, eight, then that means that it takes at least nine other shareholders to win a vote against the controlling shareholder of firm j. That is, the larger the "order" τ_j is, the 'tighter' is the grip of the controlling shareholder on company j.

Carrying this intuition further, control assignments can be distinguished according to their associated "orders of control." This is done in the following definition that singles out those with the largest τ's.

Definition 12 A control assignment C [that satisfies (10)] is *maximally stable* if there is no other solution C' to (10) such that $\tau(\Xi + C'\Sigma) > \tau(\Xi + C\Sigma)$.

Since $\{1, \ldots, n-1\}^m$ is a finite set, there exists a maximally stable control assignment for every pair of matrices Ξ and Σ. Such control assignments, therefore, appear to be the most interesting ones among the (possibly) many control assignments. Accordingly, in the empirical computations we concentrate on maximally stable control assignments.

As Remark 11 shows, the "orders" $\tau = (\tau_1, \ldots, \tau_m)$ for maximally stable control assignments also yields information on the number of controlling shareholders (positive control coefficients). There can only be one controlling shareholder if $1 < \tau_j < n-1$, and at most two if $\tau_j = n-1$. In the latter case, however, the two controlling shareholders must command 50% of the votes each. Thus, $\tau_j = 1$ is essentially the only case with multiple *simultaneously* controlling shareholders.

5 The Allianz case

In this section, we apply the technique of control assignments to a particularly prominent case of cross-ownership among firms, the conglomerate around Allianz, Münchener Rückversicherung, and E.ON AG in Germany, as of the end of 2000. By contrast to a pyramid, this is a case where a controlling shareholder cannot easily be identified by working backwards along a chain of firms. Cross-holdings in this case often go both ways and the picture resembles more a spider web than a chain: Table 1 gives the part of the cross-ownership matrix Σ that is relevant for the conglomerate around Allianz and E.ON AG. (Blank entries are zeros.)

Table 1 Cross-ownership around Allianz and E.ON

Percent	ALL	MRV	HVB	DTB	DRB	EON	EOE	EAM
ALL		24.9	13.6	4.6	21.2	10.6		
MRV	24.95		5.1	1.5		0.7		
HVB	6.77	13.3						
DTB	4.2							
DRB	10							
EON							100	
EOE			6.6					46
EAM						0.6		

ALL Allianz AG München, *MRV* Münchener Rückversicherung AG München, *HVB* Bayerische Hypo- und Vereinsbank AG München, *DTB* Deutsche Bank AG Frankfurt, *DRB* Dresdner Bank AG Frankfurt, *EON* E.ON AG Düsseldorf, *EOE* E.ON Energie AG München, *EAM* Energie-AG Mitteldeutschland EAM Kassel

The companies listed in Table 1 are only the core ones. There are 287 other firms that are connected to this network. But those are mostly controlled as subsidiaries by the companies in the core network. Therefore, these 287 companies are not represented in Table 1.

The data used for computations comprises a sample of 11,181 German firms and 13,010 ultimate shareholders, as published by Commerzbank (Commerzbank 2001) for the year 2000—a dataset that has been widely used in the literature on cross-ownership. Since the sample is not all-encompassing, of course, the set of ultimate shareholders does contain firms. Those are treated as investors in the computations, since the data does not contain information on who owns these firms. The data on ultimate shareholders also gets augmented by introducing artificial rows for dispersed undisclosed shareholders and for potential management control. Potential management control—as in the initial 'Noren Inc.' example—is allowed for by introducing an artificial row of zeros in the shareholder matrix Ξ (see Sect. 4).

Dispersed undisclosed owners are accounted for by extra rows in Ξ with appropriate small entries. If, for instance, 20% of a firm's shares are held by dispersed owners and "dispersed ownership" is defined by the data source as less than 1%, then we add twenty rows that contain zeros except for a 0.01 in the column corresponding to the firm under scrutiny. These corrections are needed to compensate for the fact that the data is necessarily partial. The picture that emerges from the numerical results is still quite clear.

Overall cross-ownership is not common in the sample, as 81% of all firms in the dataset do not hold shares in other companies. The remaining 19% (2,116 companies) can be decomposed into several blocks. The two—by far—largest such blocks are those around Allianz and E.ON AG.

The core of the first block, the "Allianz block," consists of the financial intermediaries given in the first five rows of Table 1. The core of the other one, the "E.ON block," consists of the three energy companies listed in the bottom three rows of Table 1. In all control assignments that we obtained each of the two blocks is controlled by a single investor—once even by the same. Consequently,

Table 2 Controlling shareholders for the Allianz block

Public entities
Freistaat Bayern, München
Landkreis Göttingen, Göttingen
Landkreis Northeim, Northeim
Kreis Höxter, Höxter
Main-Kinzig-Kreis, Hanau
Stadt Göttingen, Göttingen
Kreis Kassel, Kassel
Schwalm-Eder-Kreis, Homburg
Kreis Hersfeld-Rotenburg, Bad Hersfeld
Lahn-Dill-Kreis, Wetzlar
Kreis Marburg-Biedenkopf, Marburg
Werra-Meißner-Kreis, Eschwege
Kreis Heiligenstadt, Heilbad Heiligenstadt
Kreis Waldeck-Frankenberg, Waldeck
Companies
AB Industriebesitz u. Beteiligg. AG&Co.KG, München
HI-Vermögensverwaltungsges. mbH, München
FGF GmbH, Frankfurt a.M.
Vermo Vermögensverwaltungsges. mbH, Düsseldorf
AZ-AVG Beteiligungsges. mbH, München
AZ-Nona Beteiligungsges. mbH, München
BOJA Beteiligungs AG&Co., Eschborn
Zweite Herakles Beteiligungsges. mbH&Co. KG, Bad Vilbel
VI-Industrie-Beteiligungsges. mbH, München
Banque Nationale de Paris S.A.
Sydkraft AB, Malmö
Caja de Ahorros y Pensiones S.A., Barcelona
Others
Private investors (undisclosed)
Employees
Dispersed owners
Management
Pool 31
Pool 118

whoever controls one of the companies in a block controls all companies in this block.

The data set allows for 32 different control assignments. As far as the Allianz block is concerned, there are three groups of control assignments that are consistent with the data. Those are listed in Table 2. In the first group there are fourteen public sector entities that may control the Allianz block. Those are various communes and one province (Freistaat Bayern), all of which apparently hold directly or indirectly non-negligible interests in the financial intermediaries making up the Allianz block. In the second group of solutions there are twelve different companies, three of which are foreign firms, potentially controlling the Allianz block. The German companies in this group are largely holding companies, the ownership of which is not represented in the data set. Among the foreign firms two are banks, one French and the other Spanish, and one is a Swedish energy firm. The last group ("others") contains private investors, whose identity is not recorded by the data set, employees, dispersed owners (the artificial rows that account for small shareholdings), management (the artificial row of zeros), and two "pools." The latter are undisclosed groups of investors, who share profits from their holdings, as allowed for by the German tax code.[5] One of them is of particular interest: Pool 118. This pool turns out to be the controlling shareholder of the E.ON block in *all* the 32 different control assignments.

Control of any one of the two blocks implies control over large groups of other companies in all control assignments. There is one group of 151 companies that is controlled by whoever controls the Allianz block. A second group of 136 firms is controlled by the controlling shareholder of the E.ON block, i.e. by Pool 118, in all control assignments.[6]

The large number of control assignments appears discouraging at first sight, in particular as more than a third of the potentially controlling shareholders for the Allianz block turn out to be holding companies. Moreover, the presence of dispersed owners, employees, and management as potentially controlling shareholders indicates a certain fragility of control assignments. This is, however, not surprising if one considers management control as a realistic possibility. Recall that management is represented as an artificial owner, who holds no shares directly. If companies can be controlled without privately held shares, then they can also be controlled with very few shares. For this reason the vast multiplicity of solutions is to be expected.

But applying the stability criterion from Definition 12 yields an unambiguous result for control of the Allianz block: the unique maximally stable control assignment entails Pool 118 controlling the Allianz block—and the E.ON block. According to this criterion, therefore, the two big blocks are just one. In this particular solution Pool 118 controls an important part of the German financial sector and almost all of the German energy sector.[7]

If the Allianz and the E.ON block are indeed one block, a non-negligible part of the German economy was under the control of Pool 118 in the year 2000. There are 287 firms—not including the Allianz nor the E.ON block from Table 1—in our dataset for which Pool 118 obtains a control coefficient of 1 in some control assignments. (Amazingly, this is all through a single direct ownership share of 48.7% in E.ON AG, Düsseldorf, which is Pool 118's *only* direct ownership asset. Thus, whether Pool 118 or E.ON AG are viewed as the controlling shareholder in the year 2000 does not make a difference.)

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[5] There are 473 distinct pools in our data set, which makes for 3.6% of all ultimate shareholders.

[6] A complete list of these 287 firms is available from the authors upon request.

[7] Of course, we have tried to find out the identity of Pool 118. Unfortunately, neither the publisher of the data set was able to help, nor was a German economics research institution to which we turned for help.

Table 3 Control and dividend rights

Firms	Order τ	Contr. (%)	Imp. (%)
Allianz AG	42,369	45.9	0.37
Münchener Rückversicherung	23,601	38.2	0.54
Bayerische Hypo- und Vereinsbank	9,343	25.3	3.30
Deutsche Bank AG	2,200	6.1	0.02
Dresdner Bank AG	5	21.2	0.07
E.ON Energie AG München	–	100.0	48.88
E.ON AG Düsseldorf	4,604	60.6	48.88
Energie-AG Mitteldeutschland EAM	8	46.0	22.48

The "orders" τ at which the eight core firms are controlled by Pool 118 according to the maximally stable control assignment are summarized in the second column of Table 3 ("Order τ"). With the exception of Dresdner Bank all financial intermediaries are pretty tightly controlled by Pool 118. In particular, in this control assignment Pool 118 is the *only* controlling shareholder, by Remark 11. (The blank entry for E.ON Energie AG, München, indicates that this entity is 100% owned by E.ON AG, Düsseldorf.) The energy firms, in particular Energie-AG Mitteldeutschland, are less tightly controlled, according to this measure.

The third ("Contr.") resp. fourth ("Imp.") column of Table 3 give the *controlled* shares, according to $\Xi + C\Sigma$, resp. the *imputed* shares, according to $\Xi\,(I - \Sigma)^{-1}$, the former for the maximally stable control assignment (in percent). The first provides an alternative measure for how tight Pool 118's control is; the second measures dividend rights. A comparison between the third and fourth column of Table 3, therefore, provides a picture of the wedge between control and dividend rights: even though Pool 118's dividend rights in companies from the Allianz block are all below 4%, the control rights vary between 6 and 46%. For instance, in the Allianz AG Pool 118 owns only 0.37% of the dividend rights, but controls almost 46% of the voting rights, according to the maximally stable control assignment. Similarly, Pool 118 owns virtually no dividend rights in Dresdner Bank, but controls more than 20% of its voting shares.

While the maximally stable control assignment points to Pool 118 as the controlling shareholder for both the Allianz and the E.ON block, other possibilities are consistent with the data. First, the underlying data set may be insufficient to identify the ultimately controlling shareholder, as such an investor may be behind one of the holding companies in Table 2. Or the Allianz block may be controlled by a foreign firm, for instance. Given the national pride associated with the involved firms in the German public, this does not appear as the most plausible possibility, though.

More likely appears the possibility of management control. The multiplicity of solutions and that several small groups—like dispersed owners or employees—show up in control of the block, indicates that very little in privately held shares is enough to control the whole block. This points to management control of the Allianz block in our view. For, when the controlling shareholder does not obtain much in dividend rights, he may obtain his benefits from management compensation instead.

In a sense, therefore, management control appears as the second most likely solution. The remaining possibility is that the Allianz block is ultimately a public sector entity, controlled through the public entities listed in Table 2. This hypothesis, however, contrasts with the image of these firms, both in the public and at the stock market.

Even though these findings do not give an ultimate answer to “Who controls Allianz?”, the results illustrate that the present approach can identify controlling shareholders for arbitrary structures of cross-holdings. Of course, the possible multiplicity of control assignments calls for deeper empirical analysis, by comparing the boards of companies, for instance. But the computation of control assignments can act as a guide for these further considerations.

6 Conclusions and further research

A key ingredient of market economies is that the assignment of property rights implies control rights over whatever is proprietary. The term ‘private property,’ for example, is taken to mean that the owner of such property is endowed with full decision-making power over the property’s disposition, to the exclusion of non-owners. It is often argued that it is this principle that induces incentive efficiency in allocative decisions. This principle is not, however, all-encompassing, and major impediments to allocative efficiency emerge when dealing with common or shared property, rather than individually held property. Social choice paradoxes are examples of such impediments—dictatorship aside, an optimal decision mechanism among many owners may not exist.

Another problem arises when property *itself* carries property rights in other economic entities. This is the case in the fictitious ‘Noren Inc.’ example in the Introduction, in which several firms own shares in other firms. Such cross-ownership structures blur the connection between ownership and control over the disposition of a firm’s assets. As seen above, it is even possible for cross-ownership to completely divorce control from property rights.

One possible consequence of this observation is to conclude that the separation of ownership from control leaves management in control. This has become known as the hypothesis of Berle and Means (1932). Their conclusion has, however, not gone undisputed. It has been argued in Gadhoum et al. (2005), for example, that Berle and Means’ data does not actually support their hypothesis. Instead, with contemporary data, they find that almost 60% of all US listed companies have a controlling shareholder.

This paper subscribes to the approach that it should be possible to ultimately trace a controlling owner for all firms. Therefore, we address the problem of how this needs to be done in the presence of cross-ownership among firms. We provide a method how to determine the largest shareholder in terms of both direct and indirect shareholdings. Though our methodology does not necessarily yield unique control assignments for all economies, an empirical application to the conglomerate around Allianz in Germany produces reasonable information.

The present approach remains on a pure accounting level, though, as it is "preference-free." A full assessment of the implications of various control assignments would take a theoretical model of cross-ownership, in which firm investment levels and the establishment of firms are optimally selected by investors, and industry ownership structures are then determined. This is a key topic for further research.

Appendix: Proofs

Proof of Proposition 4 "if:" Suppose, first, that (a) holds. Since the status quo is accepted with certainty, no $i \in A$ with $i \neq \iota$ has an incentive to participate: if he would, the outcome would be unchanged, but he would bear the participation cost. The same applies to $i \in B$, given that $a_j = 0$ for all $j \in B$ with $j \neq i$: Since $\xi_{ij} \leq \xi_{\iota j}$, the outcome remains unchanged if i participates, but participation costs have to be privately born. Hence, (a) describes an equilibrium.

Suppose, second, that (b) holds. Since the status quo is dismissed with certainty, no $i \in B \setminus B^*$ has an incentive to participate. If some $i \in B^*$ would deviate to not participating, however, that

$$\sum_{k \in B^*} \xi_{kj} > \xi_{\iota j} \geq \sum_{k \in B^*} \xi_{kj} - \xi_{ij}$$

[from (6) and $\xi_{kj} \geq 0$ for all $k = 1, \ldots, n$] implies that the status quo would win. Thus, every $i \in B^*$ is pivotal and, therefore, optimally participates. For $i \in A$ with $i \neq \iota$ deviating to participation does not change the outcome, because

$$\sum_{h \in B^*} \xi_{hj} > \xi_{\iota j} + \xi_{kj} \quad \text{for all } k \in A \text{ with } k \neq \iota$$

[from the second inequality in (6)] so that for $i \in A$ with $i \neq \iota$ it is optimal not to participate. Hence, (b) also describes an equilibrium.

"only if:" Let $a \in \{0, 1\}^n$ with $a_\iota = 1$ for $\iota \in A$ be a pure strategy Nash equilibrium. Then either the status quo is adopted, i.e. $\sum_{i \in A} a_i \xi_{ij} \geq \sum_{i \in B} a_i \xi_{ij}$, or it is dismissed, i.e. $\sum_{i \in A} a_i \xi_{ij} < \sum_{i \in B} a_i \xi_{ij}$, with certainty.

Consider, first, the case, where it is adopted. If there were some $i \in B$ with $a_i = 1$, this i would have to be pivotal, i.e.

$$\sum_{k \in B} a_k \xi_{kj} > \sum_{k \in A} a_k \xi_{kj} \geq \sum_{k \in B \setminus \{i\}} a_k \xi_{kj} \Leftrightarrow$$
$$\xi_{ij} > \sum_{k \in A} a_k \xi_{kj} - \sum_{k \in B \setminus \{i\}} a_k \xi_{kj} \geq 0$$

which, by the hypothesis that the status quo wins, yields the contradiction

$$\xi_{ij} > \sum_{k \in A} a_k \xi_{kj} - \sum_{k \in B} a_k \xi_{kj} + \xi_{ij} \geq 0$$

to $\sum_{k\in A} a_k\xi_{kj} \geq \sum_{k\in B} a_k\xi_{kj}$. Therefore, in this case, $a_i = 0$ for all $i \in B$. Given that this is the case, $a_i = 0$ must hold for all $i \in A$ with $i \neq \iota$. Otherwise some $i \in A$ with $i \neq \iota$ could have saved on participation costs. Hence, in any pure strategy equilibrium, where the status quo wins for sure, $a_i = 0$ must hold for all $i = 1, \ldots, n$ with $i \neq \iota$. But then there cannot be any $i \in B$ with $\xi_{ij} > \xi_{\iota j}$. For, if there were, this i would be pivotal and, thus, participate in equilibrium with probability one.

Consider, second, the case, where the status quo is dismissed for sure. If there were some $i \in A$ with $i \neq \iota$ with $a_i = 1$, then this i could save on participation costs without changing the outcome. Therefore, $a_i = 0$ for all $i \in A$ with $i \neq \iota$. Let, without loss of generality, $a_i = 1$ for $i \in B^*$ and $a_i = 0$ for $i \in B \setminus B^*$ for some subset $B^* \subseteq B$, so that $\sum_{i\in B} a_i\xi_{ij} = \sum_{i\in B^*} \xi_{ij} > \sum_{i\in A} a_i\xi_{ij} = \xi_{\iota j}$. By the assumption of equilibrium, every $i \in B^*$ must be pivotal, i.e.

$$\sum_{k\in B^*} \xi_{kj} > \xi_{\iota j} \geq \sum_{k\in B^*\setminus\{i\}} \xi_{kj} = \sum_{k\in B^*} \xi_{kj} - \xi_{ij},$$

because otherwise i could improve by not participating. The second of the above inequalities implies $\xi_{ij} \geq \sum_{k\in B^*} \xi_{kj} - \xi_{\iota j}$ for all $i \in B^*$, i.e. the first part of (6). If there were some $h \in A$ with $h \neq \iota$ such that $\xi_{\iota j} + \xi_{hj} \geq \sum_{i\in B^*} \xi_{ij}$, this h would be pivotal, implying that in equilibrium $a_h = 1$ would have to hold—a contradiction. Therefore, $\sum_{i\in B^*} \xi_{ij} > \xi_{\iota j} + \xi_{hj}$ for all $h \in A$ with $h \neq \iota$, which yields the second part of (6). □

Proof of Corollary 5 (a) The condition in Proposition 4(a) must fail, because $\xi_{ij} > \xi_{\iota j}$ for some $i \in B$. So, in any pure strategy equilibrium the status quo is dismissed. Setting $B^* = \{i\}$ yields that $\xi_{ij} \geq \xi_{ij} - \xi_{\iota j} > \max_{k\in A\setminus\{\iota\}} \xi_{kj}$, verifying (6).

(b) The first condition is sufficient for the existence of a pure strategy equilibrium in which the status quo is adopted by Proposition 4(a). Since $\sum_{i\in B^*} \xi_{ij} \leq \sum_{i\in B} \xi_{ij} \leq \xi_{\iota j} + \max_{k\in A\setminus\{\iota\}} \xi_{kj}$ for all $B^* \subseteq B$, condition (6) fails. □

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